Chapter 6 Orthogonality and Least Squares

6.1 Inner Product, Length, and Orthogonality

The Inner Product

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$. The number $\mathbf{u}^T\mathbf{v}$ is called the inner product of $\mathbf{u}$ and $\mathbf{v}$, and often it is written as $\mathbf{u}\cdot\mathbf{v}$. That is, if

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

then the inner product of $\mathbf{u}$ and $\mathbf{v}$ is

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

Example 1. Let $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$. Compute the quantities:

$\mathbf{w}\cdot\mathbf{w}, \mathbf{x}\cdot\mathbf{w}, \dfrac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}$ and $\left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$.

ANS: $\vec{w}\cdot\vec{w} = \vec{w}^T\vec{w} = \begin{bmatrix} 3 & -1 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} = 9+1+25 = 35$

$\vec{x}\cdot\vec{w} = \begin{bmatrix} 6 & -2 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} = 18+2-15 = 5$

$\dfrac{\vec{x}\cdot\vec{w}}{\vec{w}\cdot\vec{w}} = \dfrac{5}{35} = \dfrac{1}{7}$

$\left(\dfrac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v} = \dfrac{-2+6}{4+9}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \dfrac{4}{13}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Theorem 1 Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then
a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
b. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
c. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$
d. $\mathbf{u} \cdot \mathbf{u} \geq 0,$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

The Length of a Vector

If $\mathbf{v}$ is in $\mathbb{R}^n$, with entries $v_1, \ldots, v_n$, then the square root of $\mathbf{v} \cdot \mathbf{v}$ is defined because $\mathbf{v} \cdot \mathbf{v}$ is nonnegative.

Definition. The **length** (or **norm**) of $\mathbf{v}$ is the nonnegative scalar $\|\mathbf{v}\|$ defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}, \quad \text{and} \quad \|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$$

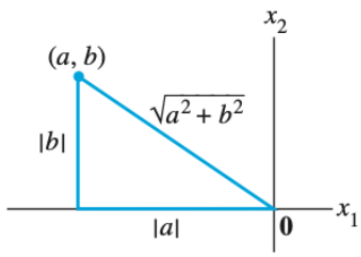


FIGURE 1
Interpretation of $\|\mathbf{v}\|$ as length.

Remarks:

1. For any scalar c, the length of $c\mathbf{v}$ is $|c|$ times the length of $\mathbf{v}$. That is,

$$\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$$

2. A vector whose length is 1 is called a **unit vector**.
3. Divide a nonzero vector $\mathbf{v}$ by its length: $\mathbf{v}/\|\mathbf{v}\|$, we get a unit vector since the length of it is $(1/\|\mathbf{v}\|)\|\mathbf{v}\| = 1$. This process is sometimes called normalizing.

Example 2. Find a unit vector in the direction of the given vector.

$$\vec{v} = \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}$$

By Remark 3. we know the unit vector in the direction of $\vec{v}$ is

$$\frac{\vec{v}}{\|\vec{v}\|} = \frac{\begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}}{\sqrt{3^2+6^2+(-3)^2}} = \frac{1}{3\sqrt{6}}\begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Distance in $\mathbb{R}^n$

Definition. For $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the distance between $\mathbf{u}$ and $\mathbf{v}$, written as $\operatorname{dist}(\mathbf{u}, \mathbf{v})$, is the length of the vector $\mathbf{u} - \mathbf{v}$. That is,

$$\operatorname{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

Example 3. Compute the distance between the vectors $\mathbf{u} = (7, 1)$ and $\mathbf{v} = (3, 2)$.

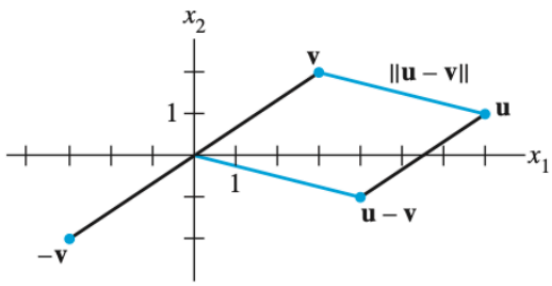


FIGURE 4 The distance between $\mathbf{u}$ and $\mathbf{v}$ is the length of $\mathbf{u} - \mathbf{v}$.

$$\operatorname{dist}(\vec{u}, \vec{v})$$
$$= \|\vec{u} - \vec{v}\|$$
$$= \|(4, -1)\|$$
$$= \sqrt{4^2 + (-1)^2}$$
$$= \sqrt{17}$$

Example 4. If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, then

$$\begin{aligned} \operatorname{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| &= \sqrt{(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})} \\ &= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2} \end{aligned}$$

Orthogonal Vectors

Definition. Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are orthogonal (to each other) if $\mathbf{u} \cdot \mathbf{v} = 0$.

Example 5. Determine which of the following pairs of vectors are orthogonal.

(1) $\mathbf{u} = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$

(2) $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix}$

(1) Compute $\vec{u} \cdot \vec{v} = 12 \times 2 - 3 \times 3 - 15 = 0$

Thus $\vec{u}, \vec{v}$ are orthogonal to each other.

(2) Compute $\vec{u} \cdot \vec{v} = -12 + 2 + 10 + 0 - 6 = 0$

Thus $\vec{u}, \vec{v}$ are orthogonal to each other.

The Pythagorean Theorem

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

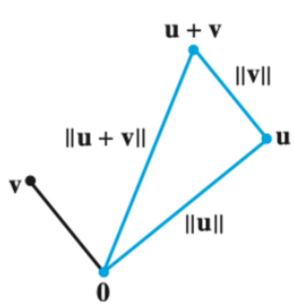


FIGURE 6

proof: $\vec{u}, \vec{v}$ are orthogonal

$\iff \vec{u} \cdot \vec{v} = 0$

$\iff \|\vec{u} + \vec{v}\|^2$

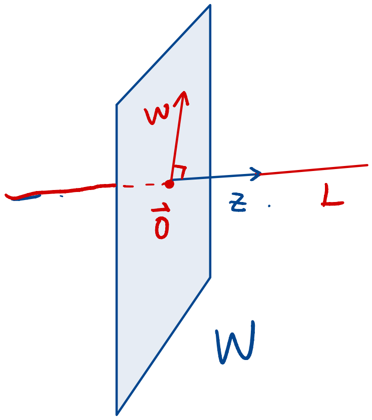
$= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})$

$= \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} + \cancelto{0}{\vec{u} \cdot \vec{v}} + \cancelto{0}{\vec{v} \cdot \vec{u}}$

$= \|\vec{u}\|^2 + \|\vec{v}\|^2$

Orthogonal Complements

If a vector $\mathbf{z}$ is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, then $\mathbf{z}$ is said to be **orthogonal to** W**.** The set of all vectors $\mathbf{z}$ that are orthogonal to W is called the **orthogonal complement** of W and is denoted by $W^\perp$ (and read as " W perpendicular" or simply " W perp").



Example: Let W be a plane through $\vec{0}$ in $\mathbb{R}^3$.

L: the line through the origin and perpendicular to W

Let $\vec{z}$ be a vector on L, $\vec{w}$ be a vector W.

Then $L = W^\perp$, $W = L^\perp$

Theorem Let W a subspace of $\mathbb{R}^n$

1. A vector $\mathbf{x}$ is in $W^\perp$ if and only if $\mathbf{x}$ is orthogonal to every vector in a set that spans W.
2. $W^\perp$ is a subspace of $\mathbb{R}^n$.

Theorem 3 Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T :

$$(\text{Row } A)^\perp = \text{Nul } A \quad \text{and} \quad (\text{Col } A)^\perp = \text{Nul } A^T$$

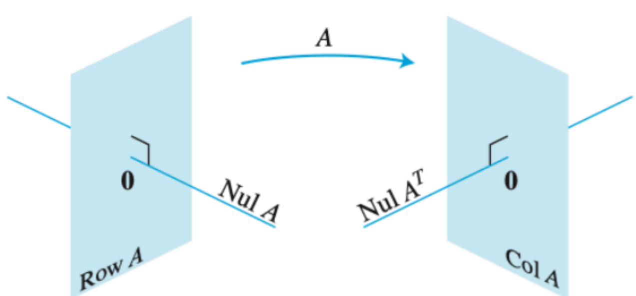


FIGURE 8 The fundamental subspaces determined by an $m \times n$ matrix A.

Exercise 6. Let $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$. Describe the set H of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ that are orthogonal to $\mathbf{v}$. [Hint: Consider $\mathbf{v} = \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$.]

Solution. When $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$, the set H of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ that are orthogonal to v is the subspace of vectors whose entries satisfy $ax + by = 0$. If $a \neq 0$, then $x = -(b/a)y$ with y a free variable, and H is a line through the origin. A natural choice for a basis for H in this case is $\left\{ \begin{bmatrix} -b \\ a \end{bmatrix} \right\}$. If $a = 0$ and $b \neq 0$, then $by = 0$. Since $b \neq 0, y = 0$ and x is a free variable. The subspace H is again a line through the origin. A natural choice for a basis for H in this case is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$, but $\left\{ \begin{bmatrix} -b \\ a \end{bmatrix} \right\}$ is still a basis for H since $a = 0$ and $b \neq 0$. If $a = 0$ and $b = 0$, then $H = \mathbb{R}^2$ since the equation $0x + 0y = 0$ places no restrictions on x or y.